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1999 J. Phys. A: Math. Gen. 32 3649

(http://iopscience.iop.org/0305-4470/32/19/314)

J. Phys. A: Math. Gen. **32** (1999) 3649–3661. Printed in the UK PII: S0305-4470(99)00328-5

Exact solutions for the general nonstationary oscillator with a singular perturbation

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Received 21 December 1998

Abstract. Three linearly independent Hermitian invariants for the nonstationary generalized singular oscillator (SO) are constructed and their complex linear combination is diagonalized. The constructed family of eigenstates contains as subsets all previously obtained solutions for the SO and includes all Robertson and Schrödinger intelligent states for the three invariants. It is shown that the constructed analogues of the $SU(1,1)$ group-related coherent states for the SO minimize the Robertson and Schrödinger uncertainty relations for the three invariants and for every pair of them simultaneously. The squeezing properties of the new states are briefly discussed.

1. Introduction

Recently considerable attention has been paid in the literature [1–5] to the (nonstationary) singular oscillator (SO), i.e. the particle with mass m in the harmonic plus an inverse harmonic potential

$$V(x) = \frac{1}{2}m\omega^2x^2 + g\frac{1}{x^2} \tag{1}$$

where the mass and/or frequency may depend on time. Previously this SO has been treated in a number of papers [6–14], exact invariants and wavefunctions being obtained for the case of stationary SO (constant m, ω and g) in [12, 14] and of SO with varying frequency $\omega(t)$ (but constant m, g) in [11, 13]. The more general cases treated in [7, 9] correspond (as in [2]) to $m(t)g(t) = \text{const}$. The potential (1) has wide applications in molecular and solid state physics: the radial motion of such systems as the hydrogen atom, the n-dimensional oscillator, the charged particle in a uniform magnetic plus electric field with scalar potential proportional to $1/(x^2 + y^2)$ and the N identical particles interacting pairwise with potential energy $V_{ij} = (m\omega^2/2)(x_i - x_j)^2 + g/(x_i - x_j)^2$ can be reduced [12, 13, 15] to the case of SO. The potential (1) was recently applied to describe a two-ion trap [1].

The aim of this paper is to construct a new family of exact wavefunctions for the SO and to extend these and the previous solutions [2, 9, 11, 12] to the case of a nonstationary general oscillator with the singular perturbation $g(t)/x^2$, $m(t)g(t) = \text{const}$ (general SO). To this aim we make an efficient use of the method of time-dependent quantum invariants [16–18]. We construct a four complex parameter (z, u, v, w) family of states $|z, u, v, w; \kappa, t\rangle$ of general SO, which diagonalize the general complex combination of the three linearly independent Hermitian invariants $I_j(t)$.

The large family $|z, u, v, w; \kappa, t\rangle$ of general SO states contains all previously obtained solutions as subsets and new states with interesting properties. In particular, it contains the analogues of the Barut–Girardello coherent states (CS) and the $SU(1, 1)$ group-related CS with symmetry [15,19]. The most important physical properties of the new wavefunctions of the SO (which the previous solutions lack) are the strong squeezing [20,21] in the $SU(1, 1)$ generators and the maximal intelligency [21, 22] with respect to Robertson [23] and Schrödinger [23] uncertainty relations. In particular, the new states can exhibit strong quadratic squeezing [24]. The obtained family of states can be regarded as an extension of the $su^C(1, 1)$ algebra-related coherent states (CS) $|z, u, v, w; k\rangle$ [21, 25, 26] from the series $D^+(k)$ with discrete Bargman index $k = \frac{1}{4}, \frac{3}{4}, k = \frac{1}{2}, 1, \ldots$ [15] to the series with continuous Bargman index $\kappa \geqslant \frac{1}{2}$. Recall that the nonstationary quadratic in the coordinate and the momentum Hamiltonian, which has wide applications in quantum optics (see [20] and references therein), is an operator from the $su(1, 1)$ representation with Bargman index $k = \frac{1}{4}, \frac{3}{4}$.

In section 2 the three independent invariants $I_j(t)$ for the general SO are constructed and expressed in terms of a complex parameter $\epsilon(t)$ which obeys the classical oscillator equation. The three invariants close the $su(1, 1)$ algebra in a continuous series representation. In section 3 the general complex combination of the invariants $I_j(t)$ is diagonalized and some limiting cases of the corresponding eigenstates $|z, u, v, w; \kappa, t\rangle$, are considered. The Green function of the general SO is also written down. The overcompleteness, intelligent and squeezing properties [20–22] of the constructed states are briefly outlined. It is shown that the wavefunctions $\Psi_\xi(x, t)$, which are the analogue of the $SU(1, 1)$ group-related CS with symmetry, are states with maximal intelligency, i.e. they simultaneously minimize the Schrödinger uncertainty relation for every three pairs of invariants $I_i(t)$, $I_j(t)$ and the Robertson inequality for the three invariants. Finally, in section 4 we give a summary and some concluding remarks.

2. Symmetry and invariants for the general SO

The Hamiltonian of the general oscillator with a singular perturbation we consider is of the form

$$H(t) = \frac{1}{2m(t)}p^2 + b(t)(px + xp) + \frac{m(t)\omega^2(t)}{2}x^2 + \frac{g(t)}{x^2} \tag{2}$$

where $m(t), \omega(t)$ and $b(t)$ are arbitrary real differentiable functions, $m(t) > 0$. The time dependence of $g(t)$ is related to that of $m(t)$ as

$$2m(t)g(t)/\hbar^2 = c = \text{const} \tag{3}$$

where c is arbitrary real (dimensionless) parameter. As we shall see below, the constraint (3) ensures $H(t)$ belongs to the algebra $su(1, 1)$. For brevity system (2) should be referred to as the general SO. Exact invariants and wavefunctions for various particular cases of (2) have been considered in the literature: $b(t) = 0$, constant m, ω and g—in [12, 14]; $b(t) = 0$, constant m, g and varying $\omega(t)$—in [11, 13]; $b(t) = 0$, varying $m(t)$, $\omega(t)$, $g(t)$ with the constraint (3) with $c = 1$—in [2]; varying $b(t)$ and $\omega(t)$ and constant m and g—in [9]. In [7] three Heisenberg operators and their correlation functions for (2) with (3) were considered (in different parametrizations). In this section we construct three linearly independent exact invariants $I_j(t)$ for the general SO (2), the invariants being expressed in terms of a complex time-dependent parameter $\epsilon(t)$ which obey the classical harmonic oscillator equation. The invariants $I_j(t)$ are represented as time-dependent linear combinations of the three $su(1, 1)$ operators L_j.

It is worth noting that the time dependence of the mass (and of the coupling $g(t)$ if (3) is valid) can be eliminated by the simple timescale transformation,

$$t \to t' = \int^t \mathrm{d}\tau\, m_0/m(\tau).$$

The resulting Hamiltonian H' is with constant mass m_0 and coupling $g_0 = mg/m_0$ and new time-dependent frequencies $\omega' = m\omega/m_0$ and $b' = mb/m_0$. For the latter Hamiltonian exact invariants, wavefunctions and Green function were constructed in [9].

The operator $px + xp$ is easily recognized as the pure squeezing generator [20]. One can check that the time-dependent canonical transformation, generated by the squeeze operator $S(\tilde{b}(t)) = \exp[(-\mathrm{i}/\hbar)\tilde{b}(t)(xp + px)]$, $\tilde{b}(t) = \int^t b(\tau)\,\mathrm{d}\tau$, converts (2) into SO Hamiltonian H' with the same frequency and new mass $m' = m\exp[-4\tilde{b}(t)]$ and new coupling $g' = g\exp[4\tilde{b}(t)]$. Time-dependent canonical transformations are in fact very powerful—they can convert a given Hamiltonian into any desired one [27]. On the classical level switching on the term $b(xp + px)$ results in a sudden change of the squared frequency from ω^2 to $\omega^2 - \omega_1^2(b)$, $\omega_1^2 = 4b^2 + 2\dot{b} + 2b\dot{m}/m$. If $\omega_1^2 > \omega^2$ one gets the inverted oscillator. This motivates the term ‘general oscillator’ for the Hamiltonian system (2) with $g = 0$.

We first construct the time-dependent invariants for the system (2). The defining equation of the invariant operators $I(t)$ for a quantum system with Hamiltonian H is

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \frac{\partial I(t)}{\partial t} - \frac{\mathrm{i}}{\hbar}[I(t), H] = 0. \tag{4}$$

Formal solutions to this equation are operators $I(t) = U(t)I(0)U^\dagger(t)$, where $U(t)$ is the evolution operator of the system, $U(t) = T\exp[-(\mathrm{i}/\hbar)\int^t H(\tau)\,\mathrm{d}\tau)]$. However, the explicit construction greatly simplifies if one can guess the operator structure of $I(t)$ and substitute it in (4). The method of time-dependent invariants was developed and efficiently used in [16–18] to construct exact wavefunctions for the varying frequency and mass oscillator [16, 17] and for n-dimensional nonstationary quadratic systems [17, 18]. In particular, the time evolution of the Glauber CS and Fock states was explicitly found for general quadratic Hamiltonians [18] (Quantum mechanical studies of quadratic Hamiltonians can be found in many later papers (see [28] and references therein).) In [7] three linearly independent Heisenberg operators for the general SO were constructed in the form of elements of the $su(1, 1)$ algebra.

We are looking for solutions of equation (4) for the general SO (2) of the same operator form as that of the Hamiltonian,

$$I(t) = \alpha(t)p^2 + \beta(t)(xp + px) + \gamma(t)x^2 + \delta(t)\frac{1}{x^2} \tag{5}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ may be real or complex functions of time (then $I(t)$ would be Hermitian or non-Hermitian invariant). Substituting (5) and (2) into (4) (and assuming $[x, p] = \mathrm{i}\hbar$) we obtain equations for the above coefficients:

$$\alpha(t)g(t) = \delta(t)/2m(t) \tag{6}$$

$$\begin{aligned}
\dot{\alpha}(t) &= 4\alpha(t)b(t) - 2\beta(t)/m(t)\\
\dot{\beta}(t) &= 2\alpha(t)m(t)\omega^2(t) - \gamma(t)/m(t)\\
\dot{\gamma}(t) &= 2[\beta(t)m(t)\omega^2(t) - 2\gamma(t)b(t)]\\
\dot{\delta}(t) &= 4[\delta(t)b(t) - \beta(t)g(t)].
\end{aligned} \tag{7}$$

From (6) and the first and the fourth equations in (7) we easily obtain constraint (3) on the time evolution of $m(t)$ and $g(t)$. Thus, nonstationary general SO invariants of the form (5) exist only under constraint (3). The Lie algebraic meaning of the latter is that it is under this constraint only when the Hamiltonian of general SO is an operator of the algebra $su(1,1)$. Indeed, the four operators p^2, $xp+px$, x^2 and $1/x^2$ do not close any algebra under commutations, but the three combinations L_i,

$$L_1 = \frac{1}{4}\left(Q^2 - P^2 - \frac{c}{Q^2}\right) \qquad L_2 = -\frac{1}{4}(QP+PQ) \qquad L_3 = \frac{1}{4}\left(Q^2 + P^2 + \frac{c}{Q^2}\right) \tag{8}$$

where c is arbitrary real constant, and Q and P are dimensionless coordinate and moment,

$$Q = x\sqrt{m_0\omega_0/\hbar} \qquad P = p/\sqrt{m_0\omega_0\hbar}$$

close the algebra $su(1,1)$ [15],

$$[L_1, L_2] = -\mathrm{i}L_3 \qquad [L_2, L_3] = \mathrm{i}L_1 \qquad [L_3, L_1] = \mathrm{i}L_2.$$

In the above m_0 and ω_0 are parameters with the dimension of mass and frequency respectively. They may be treated as the initial values of $m(t)$ and $\omega(t)$ in (2). One can easily check that $H(t)$, equation (2), is a linear combination of L_j if and only if the constraint (3) is satisfied:

$$\begin{aligned} H(t) = \hbar\omega_0 &\left[\left(\frac{m(t)\omega^2(t)}{m_0\omega_0^2} - \frac{m_0}{m(t)}\right)L_1 - 4\frac{b(t)}{\omega_0}L_2 + \left(\frac{m(t)\omega^2(t)}{m_0\omega_0^2} + \frac{m_0}{m(t)}\right)L_3\right] \\ &\equiv h_j(t)L_j. \end{aligned} \tag{9}$$

This $H(t)$ would be the general element of the algebra $su(1,1)$ if $h_j(t)$ can acquire arbitrary real values. This can be achieved if ω^2 can take any real values, not only positive ones (in order for h_3 to be arbitrary real), i.e. if the nonsingular part of $H(t)$ is the general quadratic in p, q form, including the inverted oscillator. In view of the above symmetry the general SO described by (2) and (3) can be adequately called a (general) $su(1,1)$ *SO*.

The Casimir invariant of the algebra spanned by L_j is

$$C_2 = L_3^2 - L_1^2 - L_2^2 = -\frac{3}{16} + \frac{c}{4} = \kappa(\kappa - 1) \tag{10}$$

where $\kappa = \frac{1}{2} \pm (\frac{1}{4})\sqrt{1+4c}$. The relation (10) means that the representation realized by L_j is reducible and the dynamical symmetry group [15, 18] of the nonstationary general SO with a fixed value of c in (3) is $SU(1,1)$. Note that for a given $c \equiv 2mg/\hbar^2$ there are two different values of the parameter kappa except for the case of $c = -\frac{1}{4}$. Kappa is real for $c \geqslant -\frac{1}{4}$ and complex for $c < -\frac{1}{4}$. The SO solutions of the previous publications [7–14] are expressed in terms of g or $a = (\frac{1}{2})\sqrt{1+8mg/\hbar^2}$. We find the continuous parameter κ (to be called the Bargman parameter) most convenient with regards to the maximal analogy with the solutions related to the discrete series representations $D^+(k)$ of $su(1,1)$.

The three *linearly independent invariants* $I_j(t)$ for the general SO under the constraint (3) are found in the form of the following time-dependent linear combinations of the dimensionless $su(1,1)$ operators L_j:

$$\begin{aligned} I_1(t) = 2\hbar &\left[\left(\frac{1}{m_0\omega_0}\mathrm{Re}\,\gamma(t) - m_0\omega_0\mathrm{Re}\,\alpha(t)\right)L_1 - 2\mathrm{Re}\,\beta(t)L_2\right] \\ &+2\hbar\left(\frac{1}{m_0\omega_0}\mathrm{Re}\,\gamma(t) + m_0\omega_0\mathrm{Re}\,\alpha(t)\right)L_3 \end{aligned} \tag{11}$$

$$\begin{aligned} I_2(t) = 2\hbar &\left[\left(-\frac{1}{m_0\omega_0}\mathrm{Im}\,\gamma(t) + m_0\omega_0\mathrm{Im}\,\alpha(t)\right)L_1 + 2\mathrm{Im}\,\beta(t)L_2\right] \\ &-2\hbar\left(\frac{1}{m_0\omega_0}\mathrm{Im}\,\gamma(t) + m_0\omega_0\mathrm{Im}\,\alpha(t)\right)L_3 \end{aligned} \tag{12}$$

$$I_3(t) = 8\overline{h}^2 \left[\left(-m_0\omega_0 \mathrm{Im}\,(\alpha(t)\beta^*(t)) + \frac{1}{m_0\omega_0} \mathrm{Im}\,(\gamma(t)\beta^*(t)) \right) L_1 - \mathrm{Im}\,(\alpha(t)\gamma^*(t)) L_2 \right]$$
$$+8\overline{h}^2 \left(m_0\omega_0 \mathrm{Im}\,(\alpha(t)\beta^*(t)) + \frac{1}{m_0\omega_0} \mathrm{Im}\,(\gamma(t)\beta^*(t)) \right) L_3 \qquad (13)$$

where c is arbitrary constant and $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are expressed in terms of one complex parameter $\epsilon(t)$ which obeys the classical oscillator equation,

$$\ddot{\epsilon}(t) + \Omega^2(t)\epsilon(t) = 0 \qquad (14)$$

$$\Omega^2(t) = \omega^2(t) - 2b(t)\frac{\dot{m}(t)}{m(t)} + \frac{\dot{m}^2(t)}{4m^2(t)} - \frac{\ddot{m}(t)}{2m(t)} - 4b^2(t) - 2\dot{b}(t). \qquad (15)$$

The expressions of $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ in terms of ϵ and $\dot{\epsilon}$ are

$$\alpha(t) = \frac{-1}{4\overline{h}m(t)}\epsilon^2(t) \qquad (16)$$

$$\beta(t) = \frac{-1}{4\overline{h}}\epsilon(t)\left[2b(t)\epsilon(t) - \dot{\epsilon}(t) + \frac{\dot{m}}{2m}\epsilon(t)\right] \qquad (17)$$

$$(17)\gamma(t) = -\frac{m(t)}{4\overline{h}}\left[2b(t)\epsilon(t) - \dot{\epsilon}(t) + \frac{\dot{m}}{2m}\epsilon(t)\right]^2. \qquad (18)$$

The above invariants I_j are Hermitian in the space of square integrable functions on the positive part of the real line which are vanishing at $x = 0$. Such are the wavefunctions constructed in the next section.

At $0 = \mathrm{d}m/\mathrm{d}t = \mathrm{d}g/\mathrm{d}t$ the invariants $I_3(t)$ and $I_-(t) = I_1(t) - \mathrm{i}I_2(t)$ coincide with the corresponding invariants constructed in [9] and at $b = 0 = \mathrm{d}m/\mathrm{d}t = \mathrm{d}g/\mathrm{d}t$ they recover those in [11, 13]. In [2] one Hermitian invariant ($\sim I_3(t)$) and its orthonormalized eigenfunctions have been obtained for Hamiltonian (2) with $b = 0$ and $c = 1$ (i.e. $2m(t)g(t) = \overline{h}^2$). The invariants $I_j(t)$ will obey the commutation relations of $su(1, 1)$ algebra,

$$[I_1(t), I_2(t)] = -\mathrm{i}I_3(t) \qquad [I_2(t), I_3(t)] = \mathrm{i}I_1(t) \qquad [I_3(t), I_1(t)] = \mathrm{i}I_2(t) \qquad (19)$$

if we fix the Wronskian of the solutions $\epsilon(t)$ of the auxiliary classical oscillator equation (14) as

$$\epsilon^*\dot{\epsilon} - \epsilon\dot{\epsilon}^* = 2\mathrm{i} \longleftrightarrow \epsilon(t) = |\epsilon(t)| \exp\left[\mathrm{i}\int^t \mathrm{d}\tau\, |\epsilon(\tau)|^{-2}\right]. \qquad (20)$$

The Casimir operator has the same value as in (10) (for any time t): $I_3^2(t) - I_1^2(t) - I_2^2(t) = -\frac{3}{16} + c/4$. The proper initial conditions which ensure $I_j(0) = L_j$ are ($\Omega_0 = \Omega(t = 0)$)

$$\epsilon(0) = \frac{1}{\sqrt{\Omega_0}} \qquad \dot{\epsilon}(0) = \mathrm{i}\sqrt{\Omega_0} \qquad \dot{b}(0) = \dot{m}(0) = 0 \qquad b(0) = 0. \qquad (21)$$

With these initial conditions one has: (a) $I_j(t) = U(t)L_jU^\dagger(t)$, where $U(t)$ is the evolution operator of the general SO, i.e. $I_j(0) = L_j$ (for other initial conditions $I_j(0)$ is a combination of L_k); (b) $I_3(0) = H_{\mathrm{SO}}(0)/(2\overline{h}\omega_0)$, where $H_{\mathrm{SO}}(0)$ is the stationary SO Hamiltonian with mass m_0, frequency ω_0 and $g_0 = c\overline{h}^2/2m_0$. If in (21) $b(0) \neq 0$ then $I_3(0) \neq L_3$, but it remains proportional to the initial Hamiltonian,

$$I_3(0) = U^\dagger(t)I_3(t)U(t) = H(0)/(2\overline{h}\omega_0) \qquad (22)$$

where $H(0)$ is the Hamiltonian (2) at $t = 0$ with $g(0) = \overline{h}^2c/2m_0$ (the stationary general SO Hamiltonian). In many papers (see [2, 29] and references therein) solutions to the quantum (singular) oscillator with varying mass and/or frequency are expressed in terms of other than

$\epsilon(t)$ parameter functions. Regarding analytic solutions to the classical equation (14) for time-dependent 'frequency' $\Omega(t)$, see [29, 30]. In [7] the Heisenberg operators $U^\dagger(t)L_jU(t) = \tilde\lambda_{jk}(t)L_k$ were constructed for the initial conditions $b(0) = 0 = \dot m(0) = \dot\omega(0) = \dot g(0)$, the coefficients $\tilde\lambda_{jk}(t)$ being expressed in terms of a parameter function $\tilde\epsilon(t)$ which obey a slightly different second-order equation.

3. Wavefunctions and algebra-related coherent states

Assuming $p = -\mathrm{i}\hbar\partial/\partial x$ the wavefunctions $\Psi(x,t)$ of the general SO (with the constraint (3)) should obey the differential (Schrödinger) equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{1}{2}\left[-\frac{\hbar^2}{m(t)}\frac{\partial^2}{\partial x^2} - \mathrm{i}\hbar 2b(t)\left(2x\frac{\partial}{\partial x}+1\right) + m(t)\omega^2(t)x^2 + \frac{c\hbar^2}{m(t)}\frac{1}{x^2}\right]\Psi(x,t). \tag{23}$$

We shall look for solutions to (23) in the form of eigenstates of the *complex linear combinations* of the constructed invariants $I_j(t)$. As in the particular cases of $b = 0$ [2–14] we consider the collapse-free case $1 + 8m(t)g(t)/\hbar^2 = 1 + 4c \geqslant 0$ ($c = \text{const}$) and look for wavefunctions $\Psi(x,t)$ which are vanishing at $x = 0$ (since at $x \to 0$ the potential may tend to ∞).

We first find the orthonormalized eigenstates $\Psi_n(x,t)$ of the invariant $I_3(t)$:

$$\Psi_n(x,t) = \sqrt{\frac{\Gamma(2\kappa)}{n!\Gamma(2\kappa+n)}}(I_+(t))^n\Psi_0(x,t) \tag{24}$$

where $I_+(t) = I_1(t) + \mathrm{i}I_2(t)$ and Ψ_0 is annihilated by $I_-(t) = I_1(t) - \mathrm{i}I_2(t)$: $I_-(t)\Psi_0(x,t) = 0$. In the above we have expressed the parameter c in $I_j(t)$ in terms of kappa: $c = 4\kappa(\kappa-1) + \frac{3}{4}$. Since $I_-(t)$ can (as in the particular case of $\dot m = 0 = b$ [11]) be cast in the form

$$I_-(t) = A(t)^2/2 + c\hbar^2\alpha(t)/2x^2 \tag{25}$$

where $A(t)$ is the invariant boson annihilation operator for the generalized oscillator ($g = 0$ in (2)) we put $\Psi_0(x,t) = \phi(x,t)\psi_0(x,t)$, where $\psi_0(x,t)$ is annihilated by $A(t)$ [18]. Then we easily find $\phi(x,t)$ and construct all wavefunctions $\Psi_n(x,t)$ (solutions to equation (23)),

$$\Psi_n(x,t) = \left[2\left(\frac{m(t)}{\hbar\epsilon^2(t)}\right)^{2\kappa}\frac{n!}{\Gamma(2\kappa+n)}\right]^{\frac{1}{2}} x^{2\kappa-\frac{1}{2}}\left(\frac{\epsilon^*(t)}{\epsilon(t)}\right)^n \exp\left[-\frac{\mathrm{i}}{\hbar}m(t)b(t)x^2\right]$$
$$\times\exp\left[\mathrm{i}\frac{m(t)}{2\hbar\epsilon(t)}\left(\dot\epsilon(t) - \frac{\dot m(t)}{2m(t)}\epsilon(t)\right)x^2\right]L_n^{2\kappa-1}\left(\frac{m(t)}{\hbar|\epsilon(t)|^2}x^2\right) \tag{26}$$

where $\epsilon(t)$ obey (14) and (20), $\Gamma(z)$ is the Gamma function, $L_n^d(x)$ are generalized Laguerre polynomials [31]. Since $L_n^d(x)$ and $\Gamma(z)$ are defined for $\mathrm{Re}d > -1$ and $\mathrm{Re}z > 0$ [31] our functions are square integrable (normalized) for $\mathrm{Re}\kappa > 0$. It is also convenient to use the Dirac notation $|\kappa,\kappa+n;t\rangle$ for the eigenstates of $I_3(t)$ and $|\kappa,\kappa+n\rangle$ for $|\kappa,\kappa+n;t=0\rangle$,

$$\Psi_n(x,t) = \langle x|\kappa,\kappa+n;t\rangle \qquad \Psi_n(x,0) = \langle x|\kappa,\kappa+n\rangle.$$

The eigenvalues of $I_3(t)$ are $\kappa + n$,

$$I_3(t)|\kappa,\kappa+n;t\rangle = (\kappa+n)|\kappa,\kappa+n;t\rangle. \tag{27}$$

The hermiticity of I_3 requires $\mathrm{Im}\,\kappa = 0$, which results in $\kappa > 0$. A further restriction on κ follows from the requirement $\Psi_n(x=0,t) = 0$, which is satisfied if

$$\kappa > \tfrac{1}{4}.$$

One can easily show that the latter constraint ensures the hermiticity of p and x in the space $\mathcal{H}_\kappa$ spanned by the wavefunctions Ψ_n on the positive part of the real line. The two values $\kappa = \frac{1}{4}$ and $\kappa = \frac{3}{4}$ correspond to $c = 0$ and the states (26) at $\kappa = \frac{3}{4}$ coincide (up to constant factors) with the odd Fock-type states (precisely: squeezed Fock states) of the general oscillator [17, 18]. It is worth noting that the expression (26) at $\kappa = \frac{1}{4}$ formally coincides (up to a constant factor, which originates from the change of the base coordinate space from the positive part to the whole real line) with the even Fock-type states of the general oscillator. The values $\kappa \geqslant \frac{1}{2}$ are related to c via $2\kappa = 1+(\frac{1}{2})\sqrt{1+4c}$, $c \geqslant -\frac{1}{4}$, while for those in the interval $\frac{1}{4} < \kappa < \frac{1}{2}$ one has $2\kappa = 1-(\frac{1}{2})\sqrt{1+4c}$. Hereafter, unless otherwise stated, we consider $2\kappa = 1+(\frac{1}{2})\sqrt{1+4c}$, i.e. the continuous Bargman index is $\kappa \geqslant \frac{1}{2}$.

At $t = 0$ and (21) the wavefunctions $\Psi_n(x,t)$ coincide with the eigenstates of the initial Hamiltonian $H(0)$ with the energy $2\hbar\omega(\kappa+n)$. The expression (26) for $\Psi_n(x,t)$ recovers the corresponding ones for the particular cases, considered previously [9, 11, 13]. With (21) and $b = 0 = \mathrm{d}m/\mathrm{d}t = \mathrm{d}g/\mathrm{d}t$ in (26) the wavefunctions $\Psi_n(x,t)$ coincide with those found in [11, 13]. In the particular case of $b = 0$ and $c = 1$ our $\Psi_n(x,t)$ failed to recover the wavefunction $\phi_n(q,t)$ of [2] (a certain t- and q-dependent factor is missing in the expression for $\phi_n(q,t)$).

The family of $\Psi_n(x,t)$ can be used to construct the Green function $G(x_2,t_2;x_1,t_1)$ for the general SO. From the definition

$$G(x_2,t_2;x_1,t_1) = \sum_{n=0}^{\infty} \Psi_n(x_2,t_2)\Psi_n^*(x_1,t_1)$$

and by means of the formula for the generating function of the product $L_n^\alpha(x)L_n^\alpha(y)$ of two associate Laguerre polynomials [31] one can obtain the closed expression

$$\begin{aligned}G(x_2,t_2;x_1,t_1) = {} & \frac{-\mathrm{i}\sqrt{m_1 m_2}}{\hbar\rho_1\rho_2\sin\gamma_{12}}(x_1x_2)^{1/2}\exp\left[\frac{\mathrm{i}}{2\hbar}(B^*(t_1)x_1^2 - B(t_2)x_2^2)\right] \\ & \times \exp\left[\frac{\mathrm{i}}{2\hbar}\mathrm{ctan}\gamma_{12}\left(\frac{m_1}{\rho_1^2}x_1^2 + \frac{m_2}{\rho_2^2}x_2^2\right)\right] I_{2\kappa-1}\left(\frac{-\mathrm{i}x_1x_2\sqrt{m_1m_2}}{\rho_1\rho_2\sin\gamma_{12}}\right) \end{aligned} \tag{28}$$

where

$$B(t) = m(t)[2b(t) - \dot\rho(t)/\rho(t) + \dot m(t)/2m(t)] \qquad \gamma_{12} = \int_{t_1}^{t_2} \mathrm{d}\tau/|\epsilon(\tau)|^2$$

$\rho_i = \rho(t_i) \equiv |\epsilon(t_i)|$, $m_i = m(t_i)$, $i = 1, 2$, and $I_\alpha(x)$ is the modified Bessel function of the first kind [31].

In order to construct a more general family of states of the general SO we note the action of the lowering and raising invariant operators $I_\mp(t)$ on $|\kappa,\kappa+n;t\rangle$. From the commutation relations (19) and the construction (24) it follows that

$$\begin{aligned} I_-(t)|\kappa,\kappa+n;t\rangle &= \sqrt{n(2\kappa+n-1)}|\kappa,\kappa+n-1;t\rangle \\ I_+(t)|\kappa,\kappa+n;t\rangle &= \sqrt{(n+1)(2\kappa+n)}|\kappa,\kappa+n+1;,t\rangle. \end{aligned} \tag{29}$$

Noting the analogy of (27) and (29) to the case of a discrete series representation of the algebra $su(1,1)$ [15] and the results of papers [25]† we construct the following family of solutions to the Schrödinger equation (23):

$$|z,u,v,w;\kappa,t\rangle = N\sum_{n=0}^{\infty} a_n(z,u,v,w,\kappa)|\kappa,\kappa+n;t\rangle \tag{30}$$

† In [25] and in [32] we have used the analytic Barut–Girardello representation on the complex plane in diagonalizing the general element of $su^C(1,1)$ (the discrete series $D^+(k)$). For the same diagonalizations the analytic representation on the unit disc was applied by Brif (first paper of [26]).

where N is the normalization factor, z, u, v and w are complex parameters and

$$a_n(z,u,v,w,\kappa) = \left(-\frac{l+w}{2u}\right)^n \sqrt{\frac{(2\kappa)_n}{n!}}\, {}_2F_1\left(\kappa + \frac{z}{l}, -n; 2\kappa; \frac{2l}{l+w}\right). \tag{31}$$

Here $l = \sqrt{w^2 - 4uv}$, $(a)_n$ is the Pochhammer symbol and ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function [31].

Using the actions (27), (29) and the relation (formula 2.8.40 of [31])

$$(c - 2b + bz - az)\, {}_2F_1(a,b;c;z) + b(1-z)\, {}_2F_1(a,b+1;c;z) + (b-c)\, {}_2F_1(a,b-1;c;z) = 0$$

one can verify (after somewhat tedious calculations) that the above solutions are *eigenstates of the complex combination* of the three invariant operators $I_j(t)$,

$$[uI_-(t) + vI_+(t) + wI_3(t)]|z,u,v,w;\kappa,t\rangle = z|z,u,v,w;\kappa,t\rangle. \tag{32}$$

Note that under the initial conditions (21) we have $[uL_- + vL_+ + wL_3]|z,u,v,w;\kappa,0\rangle = z|z,u,v,w;\kappa,0\rangle$, since under that conditions $I_\pm(0) = L_\pm \equiv L_1 \pm \mathrm{i}L_2$ and $I_3(0) = L_3$.

The states (30) are normalized but not orthogonal. Their scalar product for different parameters z,u,v,w and z',u',v',w' but the same real κ can be obtained (by making use of formula 2.5.1.12 of [31]) in the form

$$\langle t,\kappa;w,v,u,z|z',u',v',w';\kappa,t\rangle = N'N(1+s)^{z^*/l^* + z'/l}(1+s-s\zeta^*)^{-\kappa - z^*/l^*} \times (1+s-s\zeta')^{-\kappa - z'/l}\, {}_2F_1\left(\kappa + \frac{z^*}{l^*}, \kappa + \frac{z'}{l'}; 2\kappa; \frac{-s\zeta^*\zeta'}{(1+s-s\zeta^*)(1+s-s\zeta')}\right) \tag{33}$$

where $s = -(w^* + l^*)(w' + l')/(4u^*u')$, $\zeta = 2l/(w+l)$, $\zeta' = 2l'/(w'+l')$. The normalization factor N in (30) is $N = [\langle t,\kappa;w,v,u,z|z,u,v,w;\kappa,t\rangle]^{-1/2} = N(s,\zeta,\kappa + z/l,\kappa)$,

$$N^{-2}(s,\zeta,\kappa + z/l,\kappa) = (1+s)^{2\mathrm{Re}\,(z/l)}|(1-s-s\zeta)^{-(\kappa + z/l)}|^2 \times {}_2F_1\left(\kappa + \frac{z^*}{l^*}, \kappa + \frac{z}{l}; 2\kappa; \frac{-s|\zeta|^2}{|1+s-s\zeta|^2}\right). \tag{34}$$

The expressions (33) and (34) are correct if the parameter s is small [31], $|s| < 1$, i.e.

$$|w + \sqrt{w^2 - 4uv}| < 2|u|.$$

The limit $l = 0$ in formulae (30)–(34) can safely be taken.

In the above we have considered $u \neq 0$. For the case $u = 0$ (where (34) is meaningless) we treat the eigenvalue equation (32) separately and find the following normalized solutions $|z, u = 0, v, w;\kappa,t\rangle = |z_m, v, w;\kappa,t\rangle$:

$$|z_m,v,w;\kappa,t\rangle = C_m(v,w)\sum_{n=0}^{\infty}\frac{(-v/w)^n}{n!}\sqrt{(m+n)!(2\kappa)_{m+n}}|\kappa,\kappa + m + n;t\rangle \tag{35}$$

where z_m is the eigenvalue of $vI_-(t) + wI_3(t)$, $z_m = w(\kappa + m)$, $m = 0,1,2,\ldots$, and the normalization factor C_m reads ($(1)_m = m!$)

$$C_m(v,w) = [(1)_m(2\kappa)_m\, {}_2F_1(m+1, 2\kappa + m; 1; |v/w|^2)]^{-\frac{1}{2}}. \tag{36}$$

Let us note some important particular cases of the above states. The value $v = 0$ in (35) is admissible and it reproduces the eigenstates $|\kappa,\kappa + m;t\rangle$ of $I_3(t)$, the wavefunctions of which are given in (26). The values $v = 0 = w$ (and $u = 1$) in (30) are also admissible and in this way we obtain the eigenstates $|z;\kappa,t\rangle = |z,u=1,v=0,w=0;\kappa,t\rangle$ of the lowering invariant $I_-(t)$ as a particular case of (32):

$$|z;\kappa,t\rangle = [{}_0F_1(2\kappa;|z|^2)]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!(2\kappa)_n}} |\kappa,\kappa+n;t\rangle. \tag{37}$$

Using the generating function for the Laguerre polynomials $\sum_n^\infty z L_n^\alpha(x)/\Gamma(\alpha+n+1)) = (xz)^{-\alpha/2} e^z J_\alpha(2\sqrt{xz})$ [31] we obtain the normalized wavefunctions $\Psi_z(x,t) = \langle x|z;\kappa,t\rangle$ in the form

$$\Psi_z(x,t) = [{}_0F_1(2\kappa;|z|^2)]^{-\frac{1}{2}} z^{-\kappa+1/2} \left[\frac{2m(t)}{\hbar\epsilon^2(t)}x\right]^{\frac{1}{2}} J_{2\kappa-1}\left(\frac{2}{\epsilon(t)}\sqrt{\frac{m(t)}{\hbar}}x\right)$$
$$\times \exp\left[-\frac{\mathrm{i}m(t)}{2\hbar}\left(2b(t)+\frac{\dot m(t)}{m(t)}-\frac{\dot\epsilon(t)}{\epsilon(t)}\right)x^2 + z\frac{\epsilon^*(t}{\epsilon(t)}\right] \tag{38}$$

where $J_\alpha(x)$ is the Bessel function [31]. At $b=0$ and $\dot m = 0$ (also $z=\alpha^2/2$) our $\Psi_z(x,t)$ recover the corresponding wavefunctions for SO in [11, 13]. One can check that this continuous family is overcomplete in the space $\mathcal{H}_\kappa$ spanned by the orthonormalized wavefunctions $\Psi_n(x,t)$,

$$\int \mathrm{d}^2z\, f_1(|z|)|z;\kappa,t\rangle\langle t,\kappa;z| = \sum_{n=0}^{\infty} |\kappa;\kappa+n\rangle\langle n+\kappa,\kappa| \equiv 1_\kappa \tag{39}$$

where the weight function is

$$f_1(|z|) = \frac{2}{\pi} K_{2\kappa-1}(2|z|) I_{2\kappa-1}(2|z|)$$

$I_d(x)$ and $K_d(x)$ being the modified Bessel functions of the first and the third kind correspondingly. Therefore the states (37) can be considered as an extension of the Barut–Girardello CS [15] to the case of continuous representations realized by the invariants $I_j(t)$.

Another particular case of the general family $|z,u,v,w;\kappa,t\rangle$ to be noted is that of

$$u=\cosh^2 r \qquad v=\sinh^2 r\,e^{2\mathrm{i}\theta} \qquad w=\sinh(2r)e^{\mathrm{i}\theta} \qquad r\geqslant 0. \tag{40}$$

Under this choice $l^2 = w^2 - 4uv = 0$ and the solutions $|z,u,v,w;\kappa,t\rangle$ take the form of the Klauder–Perelomov $SU(1,1)$ group-related CS (see [19] and references therein)

$$|\zeta;\psi_0,t\rangle = S(\zeta,t)|\psi_0\rangle \qquad S(\zeta,t) = \exp(\zeta I_+(t) - \zeta^* I_-(t)) \qquad \zeta = r e^{\mathrm{i}\theta}$$

with the "fiducial" vector $|\psi_0\rangle = |z;\kappa,t\rangle$, where $|z;\kappa,t\rangle$ is the state (37). This parameter identification is based on the BCH formula

$$S(\zeta)I_-S^\dagger(\zeta) = \cosh^2 r\, I_- + e^{-2\mathrm{i}\theta}\sinh^2 r\, I_+ + e^{-\mathrm{i}\theta}\sinh 2r\, I_3.$$

If furthermore $z=0$ one gets the analogues of the $SU(1,1)$ group-related CS with maximal symmetry for the nonstationary general SO in the continuous representation, generated by the invariants $I_j(t)$ ($\xi = -\tanh|\zeta|\exp[-\mathrm{i}\theta]$, $|\xi|<1$)

$$|\xi;\kappa,t\rangle := S(\zeta,t)|\kappa,\kappa;t\rangle = (1-|\xi|^2)^\kappa \sum_{n=0}^{\infty}\sqrt{\frac{(2\kappa)_n}{n!}}\xi^n|\kappa,\kappa+n;t\rangle. \tag{41}$$

The wavefunction $\Psi_\xi(x,t) = \langle x|\xi;\kappa,t\rangle$ can be obtained by means of the generating function for the Laguerre polynomials $\sum_n \xi^n L_n^\alpha(x) = (1-\xi)^{-\alpha-1}\exp[x\xi/(\xi-1)]$ [31],

$$\Psi_\xi(x,t) = (1-|\xi|^2)^\kappa \sqrt{\frac{2}{\Gamma(2\kappa)}}\left(\frac{m(t)}{\hbar\epsilon^2(t)}\right)^\kappa \left(1-\xi\frac{\epsilon^*(t)}{\epsilon(t)}\right)^{-2\kappa} x^\kappa \exp\left[\frac{m(t)}{\hbar\epsilon(t)}\frac{x^2\xi}{\xi\epsilon^*(t)-\epsilon(t)}\right]$$
$$\times \exp\left[-\mathrm{i}\frac{m(t)}{2\hbar}\left(2b(t)+\frac{\dot m(t)}{2m(t)}-\frac{\dot\epsilon(t)}{\epsilon(t)}\right)x^2\right]. \tag{42}$$

The continuous family of states $\Psi_\xi(x,t)$ can also resolve the unity 1_κ in the space $\mathcal{H}_\kappa$,

$$1_k = \int_{\xi\in\mathbb{D}} d^2\xi \; f_2(|\xi|)|\xi;\kappa,t\rangle\langle t,\kappa;\xi| \qquad f_2(|\xi|) = \frac{2\kappa-1}{\pi}(1-|\xi|^2)^{-2} \tag{43}$$

where $\mathbb{D}$ is the unit disc in the complex plane.

At $b=0$ and $\dot{m}=0=\dot{g}$ and under substitution $z=\alpha^2/2$ our wavefunctions $\Psi_z(x,t)$ and $\Psi_\xi(x,t)$ recover those in papers [11, 13] (reproduced also in [1]). Moreover, it can be verified that at $\kappa=\frac{1}{4},\frac{3}{4}$ the wavefunctions $\langle x|z,u,v,w;\kappa,t\rangle$ recover (up to constant factors) the even and odd wavefunctions for the general oscillator.

Next we shall briefly discuss the intelligent [21, 22] and the squeezing [20, 21] properties of states $|z,u,v,w;\kappa,t\rangle$ (the term intelligent states was introduced in [22] for the states which minimize the Heisenberg relation for the spin components). For real w and $v=u^*$ the operator $uI_-(t)+u^*I_+(t)+wI_3(t)$ is Hermitian, therefore [32] the states $|z,u,u^*,w=w^*;\kappa,t\rangle$ minimize the Robertson inequality [23, 32] for the three observables I_j ($\vec{I}=(I_1,I_2,I_3)$):

$$\det\sigma(\vec{I}) = \det C(\vec{I}) \tag{44}$$

where σ is the uncertainty matrix, $\sigma=\{\sigma_{ij}\}$,

$$\sigma_{ij} = \langle I_iI_j+I_jI_i\rangle/2 - \langle I_i\rangle\langle I_j\rangle \equiv \Delta I_iI_j$$

and $C_{ij}=-\mathrm{i}\langle[I_i,I_j]\rangle/2$. The quantity $\sigma_{ii}=\Delta I_iI_i=\Delta^2I_i$ is called the variance of I_i. The Robertson relation for two observables is known as Schrödinger. The large family $|z,u,v,w;\kappa,t\rangle$ contains the full sets of I_i–I_j generalized intelligent states, which are defined [21] as states minimizing the Schrödinger inequality

$$\Delta^2I_i\Delta^2I_j - (\Delta I_iI_j)^2 \geqslant |\langle[I_i,I_j]\rangle|^2/4 \tag{45}$$

and therefore could also be called Schrödinger minimum uncertainty states or Schrödinger intelligent states. For example the states

$$|z,u,v,w=0;\kappa,t\rangle \equiv |z,u,v;\kappa,t\rangle$$

are I_1–I_2 Schrödinger intelligent states: the three second moments of I_1 and I_2 in $|z,u,v;\kappa,t\rangle$ are

$$\Delta^2I_1 = \frac{1}{2}\frac{|u-v|^2}{|u|^2-|v|^2}\langle I_3\rangle \qquad \Delta^2I_2 = \frac{1}{2}\frac{|u+v|^2}{|u|^2-|v|^2}\langle I_3\rangle \qquad \Delta I_1I_2 = \frac{\mathrm{Im}\,(u^*v)}{|u|^2-|v|^2}\langle I_3\rangle \tag{46}$$

and one can readily check that they minimize (45). The covariances of I_1 and I_3 and of I_2 and I_3 in $|z,u,v;\kappa,t\rangle$ read simply

$$\Delta I_1I_3 = \mathrm{Re}\,z/2 \qquad \Delta I_2I_3 = -\mathrm{Im}\,z/2. \tag{47}$$

The mean values $\langle(I_3(t))^m\rangle$, $m=1,2,\ldots,$ in the general state $|z,u,v,w;k,t\rangle$ can be calculated by means of the analytic formula

$$\langle(I_3)^m\rangle = N^2(s,\zeta,\kappa+z/l,\kappa)\left[\kappa+s\frac{\partial}{\partial s}\right]^m N^{-2}(s,\zeta,\kappa+z/l,\kappa) \tag{48}$$

$N(s,\zeta,\kappa+z/l,\kappa)$ being defined in equation (34). If $w=0$ then the condition $|s|<1$ results in $|v|<|u|$, which coincides with the normalizability condition for the eigenstates $|z,u,v;k\rangle$ of uK_-+vK_+ [21], where $K_\pm$ are Weyl operators of $su(1,1)$ in the discrete series $D^+(k)$.

It is remarkable that the wavefunctions $\Psi_\xi(x,t)$ minimize the Robertson relation (44) for the three invariants $I_1(t)$, $I_2(t)$, $I_3(t)$ and the Schrödinger inequality for all three pairs I_i, I_j *simultaneously*. Therefore $\Psi_\xi(x,t)$ minimize the third- and the second-order characteristic uncertainty relations [33] simultaneously. (The Robertson inequality for n observables relates the nth order characteristic coefficients of matrices σ and C. It was established recently [33] that similar inequalities hold for all the other characteristic coefficients.) These intelligent properties of $\Psi_\xi(x,t)$ can be directly checked by calculation of the first and second moments of $I_j(t)$,

$$\langle I_1\rangle = 2\kappa\frac{\mathrm{Re}\,\xi}{1-|\xi|^2} \qquad \langle I_2\rangle = -2\kappa\frac{\mathrm{Im}\,\xi}{1-|\xi|^2} \qquad \langle I_3\rangle = \kappa\frac{1+|\xi|^2}{1-|\xi|^2} \tag{49}$$

$$\Delta^2 I_1 = \frac{\kappa}{2}\frac{|1+\xi^2|^2}{(1-|\xi|^2)^2} \qquad \Delta^2 I_2 = \frac{\kappa}{2}\frac{|1-\xi^2|^2}{(1-|\xi|^2)^2} \qquad \Delta^2 I_3 = 2\kappa\frac{|\xi|^2}{(1-|\xi|^2)^2} \tag{50}$$

$$\Delta I_1 I_2 = -2\kappa\frac{\mathrm{Re}\,\xi\,\mathrm{Im}\,\xi}{(1-|\xi|^2)^2} \qquad \Delta I_1 I_3 = \kappa\,\mathrm{Re}\,\xi\frac{1+\xi^2}{(1-|\xi|^2)^2}$$
$$\Delta I_2 I_3 = -\kappa\,\mathrm{Im}\,\xi\frac{1+|\xi|^2}{(1-|\xi|^2)^2}. \tag{51}$$

So Ψ_ξ are states with *maximal characteristic intelligency*.

The analysis, similar to that for the states $|z,u,v;k\rangle$ [21,25], shows that the variance of I_1 (I_2) can tend to zero when $v\to u$ ($v\to -u$). Therefore, the SO states $|z,u,v;\kappa,t\rangle$ are *ideal I_1–I_2 squeezed states* [25]. I_1 (I_2) squeezing can also occur in states $|u, v=u, w\neq 0;\kappa,t\rangle$ ($|u,v=-u,w\neq 0;\kappa,t\rangle$), which minimize the Schrödinger relation for I_1 and I_3 (I_2 and I_3).

One sees that the above moments of $I_j(t)$ in $|z,u,v,w;\kappa,t\rangle$ are time independent in accordance with the fact that $I_j(t)$ are exact invariants of the system (2). Under the initial conditions (21) all the moments of $I_j(t)$ in $|z,u,v,w;\kappa,t\rangle$ coincide with those of L_j, equation (8), in $|z,u,v,w;\kappa,t=0\rangle \equiv |z,u,v,w;\kappa\rangle$. The three second moments of L_j in $|z,u,v;\kappa\rangle$ are given by the same formulae (46) with I_j replaced by L_j. So the states $|z,u,v;\kappa\rangle$ are a L_1–L_2 ideal squeezed state. The time evolution of the moments of L_j can be obtained by expressing L_j in terms of the invariants $I_j(t)$: $L_j = \Lambda^{-1}_{jk}(t)I_k(t)$. The coefficients $\lambda_{jk}(t)$ can be easily calculated from (11)–(13). The matrix $\Lambda(t)$ takes the form

$$\Lambda = 8\hbar^2\begin{pmatrix} \frac{1}{4\hbar}\left(\frac{\mathrm{Re}\,\gamma}{m_0\omega_0} - m_0\omega_0\mathrm{Re}\,\alpha\right) & \frac{1}{4\hbar}\left(\frac{\mathrm{Re}\,\gamma}{m_0\omega_0} - \mathrm{Re}\,\beta\right) & \frac{1}{4\hbar}m_0\omega_0\mathrm{Re}\,\alpha \\ \frac{1}{4\hbar}\left(\frac{\mathrm{Re}\,\gamma}{m_0\omega_0} - m_0\omega_0\mathrm{Im}\,\alpha\right) & \frac{1}{4\hbar}\left(\frac{\mathrm{Re}\,\gamma}{m_0\omega_0} - \mathrm{Im}\,\beta\right) & \frac{1}{4\hbar}m_0\omega_0\mathrm{Im}\,\alpha \\ \frac{\mathrm{Im}\,(\beta\gamma)}{m_0\omega_0} - m_0\omega_0\mathrm{Im}\,(\alpha\beta^*) & \frac{\mathrm{Im}\,(\beta\gamma)}{m_0\omega_0} - \mathrm{Im}\,(\alpha\gamma^*) & m_0\omega_0\mathrm{Im}\,(\alpha\beta^*) \end{pmatrix}. \tag{52}$$

Then the second moments $\sigma_{ij}(\vec{L})$ of L_j in any state are simply related to those of $I_j(t)$ [25,32]:

$$\sigma_{ij}(\vec{L}) = \lambda_{in}(t)\sigma_{nm}(\vec{I})\lambda_{jm}(t). \tag{53}$$

4. Concluding remarks

We have constructed three linearly independent invariants $I_j(t)$ for the general nonstationary SO (2) with (3) (the $su(1,1)$ SO) and diagonalized their general complex combination $(u+v)I_1 + \mathrm{i}(v-u)I_2 + wI_3$. The initial conditions under which $I_j(0)$ coincide with the familiar $su(1,1)$ operators L_j, equation (8), are those of equation (21). In several particular cases the closed expressions for the wavefunctions in coordinate representation is provided. The general family of the diagonalizing states $|z,u,v,w;\kappa,t\rangle$, equation (30), recovers all states previously constructed and contains all states which minimize the Robertson uncertainty relation for three observables $I_{1,2,3}(t)$ and all states which minimize the Schrödinger relation for any pair of observables $I_j(t)$, $I_k(t)$. It is established that the states $\Psi_\xi(x,t)$, equation (42), are with maximal intelligency, minimizing the Robertson relation for the three invariants $I_j(t)$ and the

Schrödinger inequality for all pairs $I_j(t)$, $I_k(t)$ *simultaneously*. Therefore, $\Psi_\xi(x,t)$ minimize the third- and the second-order characteristic uncertainty relations, established in [33].

If the singular perturbation in (2) is switched off ($g = 0 \rightarrow \kappa = \frac{1}{4}, \frac{3}{4}$), the wavefunctions $\langle x|z, u, v, w; \kappa = \frac{1}{4}, \frac{3}{4}, t\rangle$ can reproduce (up to constant factors) the corresponding time-evolved even and odd states for the general quadratic Hamiltonian [25, 26], various particular cases of which are widely discussed in the literature [20, 29].

Since the invariants $I_j(t)$ close the Lie algebra $su(1,1)$ in the continuous representation (10) the family of the diagonalizing states $|z, u, v, w; \kappa, t\rangle$, equation (30), may be called $su^C(1,1)$ algebra-related coherent states [25] for the SO. Many of the formulae concerning the SO states $|z, u, v, w; \kappa, t\rangle$ (first and second moment formulae, scalar products, resolution of unity) remain valid for the discrete series $D^+(k)$ of $su(1,1)$ by fixing the continuous parameter kappa κ equal to the discrete Bargman index $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ or $k = \frac{1}{4}, \frac{3}{4}$. It is worth recalling here that $D^+(k)$ of $SU(1,1)$, $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, has the important realization in the space of states of two mode boson/photon systems with a fixed difference of numbers of bosons/photons in the two modes. The $su^C(1,1)$ algebra eigenstates for $D^+(k)$ have been studied in detail in [21, 25, 26].

Acknowledgments

This work is partially supported by the Bulgarian Science Foundation grants No F-559 and F-644. The author thanks V V Dodonov for valuable remarks.

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